A Conformal Field Theory of Extrinsic Geometry of 2-d Surfaces *

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In the description of the extrinsic geometry of the string world sheet regarded as a conformal immersion of a 2-d surface in R^3 it was previously shown that, restricting to surfaces with $h\sqrt{g} = 1$, where h is the mean scalar curvature and g is the determinant of the induced metric on the surface, leads to Virasaro symmetry. An explicit form of the effective action on such surfaces is constructed in this article, which is the extrinsic curvature analog of the WZNW action. This action turns our to be the gauge invariant combination of the actions encountered in 2-d intrinsic gravity theory in light-cone gauge and the geometric action appearing in the quantization of the Virasaro group. This action, besides exhibiting Virasaro symmetry in z-sector, has $SL(2, C)$ conserved currents in the $\bar{z}$-sector. This allows us to quantize this theory in the $\bar{z}$-sector along the lines of the WZNW model. The quantum theory on $h\sqrt{g} = 1$ surfaces in R^3 is shown to be in the same universality class as the intrinsic 2-d gravity theory.

1. Introduction

Extrinsic geometry of surfaces immersed in R^n plays an important role in QCD-strings, 3-dimensional Ising model and in the study of biological membranes [1]. We have in our earlier publications [2, 3] approached this problem in terms of the Grassmannian σ-model. In this approach, we make use of the Gauss map [4] of conformally immersed surfaces in R^n into the Grassmannian manifold $G_{2,n}$ realized as a quadric in CP^{n-1}. This realization plays a key role in the Gauss map. However, in order for the Grassmannian manifold to form tangent planes to a given surface in R^n, $(n-2)$ integrability conditions must be satisfied by the $G_{2,n}$ fields. These conditions have been explicitly derived in [4] for R^3 and R^4 and by us [3] for R^n $(n > 4)$. The advantage of this approach to the string theory is that one is able to rewrite the string action consisting of both the Nambu-Goto and extrinsic curvature terms in terms of constrained Kähler σ model action. In a subsequent work [5] we have shown that in the geometry of surfaces immersed in R^3 with the property $h\sqrt{g} = 1$, where h is the scalar mean curvature of and g the determinant of the induced metric on the surface, there is a hidden Virasaro symmetry.

Specifically, it was shown that $H_{zz}/\sqrt{g}$, where $H_{\alpha\beta}$ are the components of the second fundamental form of the surface, transforms like energy momentum tensor (T_{zz}) and that $H_{\bar{z}\bar{z}}$ transforms like the metric tensor under the Virasaro transformation. It was further shown that in R^4 one has analogous properties for surfaces with $h\sqrt{g} = 1$, where $h = \sqrt{(h_1^2 + h_2^2)}$, and h_1 and h_2 are the two scalar mean curvatures along the two normals to the surface. The hidden symmetry in R^4 is found to be Virasaro $\otimes$ Virasaro. A simple way to see why one has doubling of the Virasaro algebra in R^4 is to observe that $G_{2,4} \simeq CP^1 \otimes CP^1$, while $G_{2,3} \simeq CP^1$. For surfaces immersed in $R^n (n > 4)$ not much is known although it is expected that one would find W-gravities in certain type of surfaces [6].

We continued our investigation of $h\sqrt{g} = 1$ surfaces in R^3 and constructed an action on such surfaces involving their extrinsic geometric properties [7] by coupling 2-d fermions to *extrinsic* curvature of the surface on which the fermions live, by minimal coupling. The minimal coupling for immersion in R^3, is through $SO(3)$ gauge fields determined by a local orthonormal frame on the surface. Integrating out the fermions [8], we obtain the WZNW action which in this case may be termed as *the induced extrinsic gravity action*. This action has local $SO(3)$ gauge invariance. The restriction to $h\sqrt{g} = 1$ partially fixes the gauge for A_z^a, $(a = \pm, 0)$ as the gauge fields encode here the information about the extrinsic properties of the immersed surface. Let us note that this restriction is not generally covariant and clearly it is a gauge choice. It

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is equivalent to an algebraic gauge condition. Our gauge choice is made in surfaces realized in a conformal immersion. In [9], the property of diffeomorphisms arising from partial fixing of $SL(2, R)$ gauge fields was considered. Here, we have found a *physical system where these gauge fields and the partial gauge fixing which were ad hoc in* [9], *arise naturally when treating* $h\sqrt{g} = 1$ *surfaces*. The resulting action is given explicitly by

$$\Gamma_{\text{eff}}(F_1, F_2) = \Gamma_+(F_2) + \Gamma_-(F_1)$$
$$- \frac{k}{4\pi} \int \left(\frac{\partial_{\bar{z}} F_2}{\partial_z F_2}\right) D_z F_1 \, dz \wedge d\bar{z} \,, \quad (1)$$

where,

$$D_z F_1 = \frac{\partial_z^3 F_1}{\partial_z F_1} - \frac{3}{2} \left(\frac{\partial_z^2 F_1}{\partial_z F_1}\right)^2 \quad (2)$$

is the Schwarz derivative of F_1.

In (1), $\Gamma_+(F_2)$ is precisely the action that appears in intrinsic 2-d gravity theory in l.c gauge [10], while $\Gamma_-(F_1)$ is the geometric action that one encounters in the quantization of the Virasaro group by the method of coadjoint orbits [11]. The coupling term in (1) is needed to make $\Gamma_{\text{eff}}(F_1, F_2)$ invariant under Virasaro transformation of F_1 and F_2, which are related to the second fundamental form as follows;

$$\frac{\partial_{\bar{z}} F_2}{\partial_z F_2} = H_{\bar{z}\bar{z}}, \quad (3)$$

and

$$D_z F_1 = H_{zz}/\sqrt{g}. \quad (4)$$

We find that (1) is invariant under Virasaro transformation, whose infinitesimal version reads as

$$\delta_{\varepsilon(\bar{z}, z)}(H_{zz}/\sqrt{g}) = -\frac{k}{2} \partial_z^3 \varepsilon(\bar{z}, z) - 2(\partial_z \varepsilon(\bar{z}, z)) H_{zz}/\sqrt{g}$$
$$- \varepsilon(\bar{z}, z) \partial_z (H_{zz}/\sqrt{g}), \quad (5)$$

and

$$\delta_{\varepsilon(\bar{z}, z)}(H_{\bar{z}\bar{z}}) = \frac{k}{2} \partial_{\bar{z}} \varepsilon(\bar{z}, z) + \varepsilon(\bar{z}, z) \partial_z H_{\bar{z}\bar{z}} - (\partial_z \varepsilon) H_{\bar{z}\bar{z}}. \quad (6)$$

Transformations (5) and (6) are equivalent to the following changes in F_i,

$$\delta_\varepsilon F_i = \varepsilon(\bar{z}, z) \partial_z F_i; \quad (i = 1, 2). \quad (7)$$

In what follows we shall call $\Gamma_+(F_2)$ as the l.c action, while $\Gamma_-(F_1)$ as the geometric action. It should be emphasized here (to avoid any confusion) that we are

discussing a 2-d *extrinsic* gravity theory (extrinsic curvature induced 2-d gravity theory) and F_1 and F_2 are related to extrinsic geometry by (3) and (4), while the corresponding action in 2-d intrinsic gravity theory is in terms of an F which parameterizes the metric tensor of the 2-d surface.

The equation of motion following from (1) is given by

$$\partial_z^3 H_{\bar{z}\bar{z}} = [\partial_{\bar{z}} - H_{\bar{z}\bar{z}} \partial_z - 2(\partial_z H_{\bar{z}\bar{z}})] \frac{H_{zz}}{\sqrt{g}}. \quad (8)$$

This equation is identically satisfied if we take both the first fundamental form (induced metric $g_{\alpha\beta}$) and the second fundamental form ($H_{\alpha\beta}$) as determined by $X^\mu(\bar{z}, z)$. However, we will see that it is convenient to regard $H_{zz}/\sqrt{g}$ and $H_{\bar{z}\bar{z}}$ as independent fields, independent of X^μ, but are related by the equation of motion (8). It is further shown that there is a useful composition formula for (1), which enables us to write it either as l.c action for $F_2 \circ F_1^{-1} = F_2(\bar{z}, F_1^{-1}(\bar{z}, z))$, where the inverse function is defined through

$$F_i(\bar{z}, F_i^{-1}(\bar{z}, z)) \equiv z, \quad (9)$$

or as the geometric action for $F_1 \circ F_2^{-1}$. This important correspondence enables us to find the conserved $SL(2, C)$ currents for the action (1) in the $\bar{z}$ (left) sector of the theory. The total energy momentum tensor $T_{\bar{z}\bar{z}}^{\text{tot}}$ is of the Sugawara form. The theory of extrinsic geometry induced 2-d gravity action for $h\sqrt{g} = 1$ surfaces in R^3 is then quantized in the left sector. From these observations, it is clear that the quantum theory of the left sector on $h\sqrt{g} = 1$ surfaces is in the same universality class as the 2-d intrinsic gravity theory.

It is well known that the conventional extrinsic curvature action [1] is given by

$$S_P = \frac{1}{\alpha_0^2} \int |H|^2 \sqrt{g} \, dz \wedge d\bar{z}. \quad (10)$$

It is interesting to note that the third term (interaction term) in (1) is precisely (10). Thus we could say that (1) is the conformally invariant extension of (10) where $H_{zz}/\sqrt{g}$ and $H_{\bar{z}\bar{z}}$ are the dynamical fields.

F_1 and F_2 appearing in (1) are the inhomogeneous coordinates of $G_{2,3} \simeq CP^1$ i.e., the Grassmannian σ-model fields arising in the Gauss map.

2. Induced Extrinsic Gravity Action

We couple 2-d fermions in the vector representation of $SO(3)$ to the gauge fields $A_{\bar{z}}, A_z$ constructed in [5].

It is well known that upon integrating out the fermions, one gets the WZNW action. We next evaluate this action explicitly in terms of F_1 and F_2 for the partially gauge fixed A_z given in [5], i.e., on $h\sqrt{g} = 1$ surfaces. Such an action was derived earlier in [9] in a different context. For this reason, we shall be brief in our adaptation to $h\sqrt{g} = 1$ surfaces.

The gauge invariant action Γ_{eff} depends on A_z and $A_{\bar{z}}$ through the parameterizations $A_z = h^{-1}\partial_z h$ and $A_{\bar{z}} = g^{-1}\partial_{\bar{z}}g$; $h, g \in SO(3)$ (h and g here are not to be confused with mean curvature scalar and determinant of the induced metric). Note that choosing h and g as independent elements of the gauge is consistent with taking $H_{\bar{z}\bar{z}}$ and $H_{zz}/\sqrt{g}$ as independent of X^μ. Explicitly

$$\Gamma_{\text{eff}} = \Gamma_-(A_z) + \Gamma_+(A_{\bar{z}}) - \frac{k}{4\pi}\operatorname{Tr}\int A_z A_{\bar{z}}\,dz \wedge d\bar{z}, \quad (11)$$

where $k = n_f$ is the number of fermions and

$$\Gamma_-(A_z) = \frac{k}{8\pi}\operatorname{Tr}\int [(\partial_z h)h^{-1}(\partial_{\bar{z}}h)h^{-1}]\,d^2\xi$$
$$+ \frac{k}{12\pi}\operatorname{Tr}\int \varepsilon^{abc}((\partial_a h)h^{-1}(\partial_b h)h^{-1}(\partial_c h)h^{-1})\,d^2\xi\,dt, \quad (12)$$

and $\Gamma_+(A_{\bar{z}})$ is given by an expression similar to (12) with h replaced by g and the sign of the WZ term changed. Gauge invariance of Γ_{eff} is ensured by the last term in (11).

Next we apply the gauge restriction to (12).

$$A_{\bar{z}}^- \equiv h\sqrt{g} = 1; \quad A_z^0 = 0. \quad (13)$$

A straightforward calculation yields

$$\Gamma_-(A_{\bar{z}}^- = 1, A_z^0 = 0, A_z^+ = H_{zz}/\sqrt{g}) \equiv S_-(F_1)$$
$$= \frac{k}{8\pi}\int\left(\frac{\partial_{\bar{z}}F_1}{\partial_z F_1}\right)\left(\frac{\partial_z^3 F_1}{\partial_z F_1} - 2\left(\frac{\partial_z^2 F_1}{\partial_z F_1}\right)^2\right)dz \wedge d\bar{z}, \quad (14)$$

where we have used (4) to write $H_{zz}/\sqrt{g}$ in terms of F_1. (14) is the geometric action for the Virasaro group studied in [11]. It is not so easy to find $\Gamma_+(A_{\bar{z}})$. However, we can evaluate it explicitly in a light-cone like gauge. Let us define the quantum action $S_+(H_{\bar{z}\bar{z}})$ by

$$\exp(-S_+(H_{\bar{z}\bar{z}})) = \int [dA_{\bar{z}}]\,\delta(A_{\bar{z}}^- - H_{\bar{z}\bar{z}})$$
$$\cdot \exp(-\Gamma_+(A_{\bar{z}}) - \int A_{\bar{z}}^+)). \quad (15)$$

Evaluating (15) at the stationary path

$$\frac{\delta\Gamma_+}{\delta A_{\bar{z}}^0} = J_z^0 = 0; \quad \frac{\delta\Gamma_+}{\delta A_{\bar{z}}^+} = J_z^- = 1, \quad (16)$$

we obtain the classical limit which can be explicitly evaluated [9] as

$$(17)$$
$$S_+(F_2) = -\frac{k}{8\pi}\int\frac{\partial_z^2 F_2}{\partial_z F_2}\left(\frac{\partial_z \partial_{\bar{z}}F_2}{\partial_z F_2} - \frac{\partial_z^2 F_2}{\partial_z F_2}\cdot\frac{\partial_{\bar{z}}F_2}{\partial_z F_2}\right)dz \wedge d\bar{z}.$$

(17) is the gauge fixed form of $\Gamma_+(A_{\bar{z}})$ in the gauge determined by (16). Note that it is analogous to the light-cone gauge fixing in 2-d intrinsic gravity theory. (17) is precisely of the form of the light-cone action in 2-d intrinsic gravity theory.

The variations of $S_-(F_1)$ and $S_+(F_2)$ under infinitesimal changes (in F_1, F_2) are given by

$$\delta S_- = -\frac{k}{4\pi}\int(\delta_\varepsilon F_1)\frac{\partial_{\bar{z}}(D_z F_1)}{\partial_z F_1}\,dz \wedge d\bar{z}, \quad (18)$$

and

$$\delta S_+ = -\frac{k}{4\pi}\int(\delta_\varepsilon F_2)\frac{1}{\partial_z F_2}\cdot\partial_z^3\left(\frac{\partial_{\bar{z}}F_2}{\partial_z F_2}\right)dz \wedge d\bar{z}. \quad (19)$$

The third term on the right hand side of (11) in the gauge $A_z^0 = 0$, $A_{\bar{z}}^- = 1$ becomes $\int(A_z^+ A_{\bar{z}}^- + A_z^+ A_{\bar{z}}^+)\,d^2\xi$, of which the second term has been taken into account in (15). The remaining term, i.e., $\int A_z^+ A_{\bar{z}}^-$, with the identification $A_z^+ = H_{zz}/\sqrt{g} = D_z F_1$ and $A_{\bar{z}}^- = H_{\bar{z}\bar{z}}$ $= \partial_{\bar{z}}F_2/\partial_z F_2$ yields $\frac{k}{4\pi}\int(D_z F_1)\left(\frac{\partial_{\bar{z}}F_2}{\partial_z F_2}\right)$ and so the total action on $h\sqrt{g} = 1$ surfaces reads

$$\Gamma_{\text{eff}}(F_1, F_2) = \frac{k}{8\pi}\int\left(\frac{\partial_{\bar{z}}F_1}{\partial_z F_1}\right)\left(\frac{\partial_z^3 F_1}{\partial_z F_1} - 2\left(\frac{\partial_z^2 F_1}{\partial_z F_1}\right)^2\right)dz \wedge d\bar{z}$$
$$- \frac{k}{8\pi}\int\frac{\partial_z^2 F_2}{\partial_z F_2}\left(\frac{\partial_z \partial_{\bar{z}}F_2}{\partial_z F_2} - \frac{\partial_z^2 F_2}{\partial_z F_2}\cdot\frac{\partial_{\bar{z}}F_2}{\partial_z F_2}\right)dz \wedge d\bar{z}$$
$$- \frac{k}{4\pi}\int\frac{\partial_{\bar{z}}F_2}{\partial_z F_2}D_z F_1\,dz \wedge d\bar{z}. \quad (20)$$

We have thus derived the extrinsic geometric gravitational WZNW action on $h\sqrt{g} = 1$ surfaces in the light-cone like gauge (16). The induced extrinsic 2-d gravity action combines in a gauge invariant way the geometric and light-cone actions which have been studied in the context of 2-d intrinsic gravity. It is remarkable that a gauge invariant interacting theory of these two actions has physical implication in the theory on $h\sqrt{g} = 1$.

Note that restricted gauge transformations have yielded diffeomorphism due to which A_z^+ which initially had conformal spin 1, has become $H_{zz}/\sqrt{g}$ $= D_z F_1$, a conformal spin 2 object. Same remark

applies to $A_{\bar{z}}^{-} \rightarrow \dfrac{\partial_{\bar{z}} F_2}{\partial_z F_2}$, which acquires a conformal spin 2. This suggests that F_2 should be regarded as a primary conformal field with conformal weight 0 in $\bar{z}$ sector. Later, we show that quantum corrections modify this property of F_2.

3. Symmetry Properties of Γ_{eff}

In this section we discuss the symmetry properties of the gauge fixed action (20) on $h\sqrt{g} = 1$ surfaces in R^3. The equation of motion for (20) is

$$\partial_z^3 \left(\frac{\partial_{\bar{z}} F_2}{\partial_z F_2} \right) - \partial_{\bar{z}} D_z F_1 - \left(\frac{\partial_{\bar{z}} F_2}{\partial_z F_2} \right) \partial_z D_z F_1$$
$$- 2 \partial_z \left(\frac{\partial_{\bar{z}} F_2}{\partial_z F_2} \right) D_z F_1 = 0 . \quad (21)$$

Using (3) and (4), it is readily seen that (21) is equivalent to (8) which we referred to as the anomaly equation. It should be mentioned here that varying (20) with respect to F_1 and F_2 independently yielded the single equation of motion (21). Our objective is to construct an effective action in terms of $H_{\bar{z}\bar{z}}$ and $H_{zz}/\sqrt{g}$ through their parameterization (3) and (4) such that these fields are independent of X^μ.

Let us first verify that (20) is invariant under infinitesimal Virasaro transformations in [5]. It can be easily verified that these transformations for $H_{\bar{z}\bar{z}}$ and $H_{zz}/\sqrt{g}$ are equivalent to the following transformation for F_i $(i = 1, 2)$.

$$\delta_{\varepsilon^-} F_i = \varepsilon^- (\bar{z}, z) \partial_z F_i, \quad (i = 1, 2) . \quad (22)$$

Invariance of (20) under (22) may be verified readily using (18) and (19). Now (22) is the infinitesimal version of the finite transformations;

$$F_1 (\bar{z}, z) \rightarrow F_1 (\bar{z}, f (\bar{z}, z)) \equiv F_1 \circ f ,$$
$$F_2 (\bar{z}, z) \rightarrow F_2 (\bar{z}, f (\bar{z}, z)) \equiv F_2 \circ f . \quad (23)$$

The infinitesimal version (22) corresponds to

$$f (\bar{z}, z) = z + \varepsilon^- (\bar{z}, z) . \quad (24)$$

We therefore conclude that

$$\Gamma_{\text{eff}} (F_1, F_2) = \Gamma_{\text{eff}} (F_1 \circ f, F_2 \circ f) . \quad (25)$$

Equation (25) has an interesting consequence. If we choose $f = F_2^{-1} (\bar{z}, z)$, where the inverse function is defined by the relation

$$F_2 (\bar{z}, F_2^{-1} (\bar{z}, z)) = z , \quad (26)$$

we find that

$$\Gamma_{\text{eff}} (F_1, F_2) = \Gamma_- (F_1 \circ F_2^{-1}) = \Gamma_+ (F_2 \circ F_1^{-1}) , \quad (27)$$

where the last equality follows by the interchange, $F_1 \leftrightarrow F_2$. We thus have derived the following useful composition formula.

$$\Gamma_+ (F_2 \circ F_1^{-1}) = \Gamma_- (F_1 \circ F_2^{-1})$$
$$= \Gamma_+ (F_2) + \Gamma_- (F_1) - \frac{k}{4\pi} \int \frac{\partial_{\bar{z}} F_2}{\partial_z F_2} D_z F_1 . \quad (28)$$

From (27) we conclude that the properties of the induced extrinsic gravity action can be understood from those of *either* the light-cone action $\Gamma_+ (F_2 \circ F_1^{-1})$ encountered in 2-d (intrinsic) gravity theory with the replacement of f of intrinsic 2-d gravity theory by $F_2 \circ F_1^{-1}$ *or* from those of $\Gamma_- (F_1 \circ F_2^{-1})$ i.e., the geometric action which arises in the quantization of the Virasaro group by coadjoint orbits (with the replacement of f by $F_1 \circ F_2^{-1}$).

The equation of motion for the light-cone action for $F_2 \circ F_1^{-1}$ is [7]

$$\partial_z^3 \left(\frac{\partial_{\bar{z}} (F_2 \circ F_1^{-1})}{\partial_z (F_2 \circ F_1^{-1})} \right) = 0 . \quad (29)$$

By using the rules for derivatives of the *composed function* $F_2 \circ F_1^{-1}$ with respect to z and $\bar{z}$, in terms of those of F_1 and F_2, it is readily verified that (29) reduces to the equation of motion (21). From (29), we conclude that $\Gamma_{\text{eff}} (F_1, F_2)$ has a hidden $SL(2, C)$ [at the algebra level it is same as $SL(2, R)$] symmetry in the $\bar{z}$-sector as in the intrinsic 2-d gravity theory. Recall that the $SL(2, C)$ currents $J_{\bar{z}}^a (a = \pm, 0)$ are defined through

$$\frac{\partial_{\bar{z}} (F_2 \circ F_1^{-1})}{\partial_z (F_2 \circ F_1^{-1})} = J_{\bar{z}}^{(+)} (\bar{z}) - 2 z J_{\bar{z}}^0 + z^2 J_{\bar{z}}^- (\bar{z}) , \quad (30)$$

where the currents are functions of $\bar{z}$ only. A straightforward calculation yields the following expressions for the currents.

$$J_{\bar{z}}^+ = - \frac{k}{2} \left[(\partial_z F_1) \left(\frac{\partial_{\bar{z}} F_2}{\partial_z F_2} \right) - \partial_{\bar{z}} F_1 \right]$$
$$- 2 F_1 J_{\bar{z}}^0 + F_1^2 J_{\bar{z}}^- , \quad (31)$$

$$J_{\bar{z}}^0 = - \frac{k}{2} F_1 J_{\bar{z}}^- + \frac{k}{2} \left[\frac{\partial_z \partial_{\bar{z}} F_2}{\partial_z F_2} - \left(\frac{\partial_{\bar{z}} F_2}{\partial_z F_2} \right) \frac{\partial_z^2 F_2}{\partial_z F_2} \right.$$
$$\left. - \frac{\partial_z \partial_{\bar{z}} F_1}{\partial_z F_1} + \frac{\partial_{\bar{z}} F_2}{\partial_z F_2} \cdot \frac{\partial_z^2 F_1}{\partial_z F_1} \right] , \quad (32)$$

and

$$J_{\bar{z}}^- = -\frac{k}{2}\left(\frac{\partial_z^2 F_1 \cdot \partial_z \partial_{\bar{z}} F_1}{(\partial_z F_1)^3} - \frac{\partial_{\bar{z}} \partial_z^2 F_1}{(\partial_z F_1)^2}\right)$$
$$-\frac{k}{2(\partial_z F_1)}\left[\partial_z^2\left(\frac{\partial_{\bar{z}} F_2}{\partial_z F_2}\right) + \left(\frac{\partial_z^2 F_1}{\partial_z F_1}\right)\partial_z\left(\frac{\partial_{\bar{z}} F_2}{\partial_z F_2}\right)\right.$$
$$\left. + \frac{\partial_{\bar{z}} F_2}{\partial_z F_2}\left[\frac{\partial_z^3 F_1}{\partial_z F_1} - \frac{(\partial_z^2 F_1)^2}{(\partial_z F_1)^2}\right]\right]. \tag{33}$$

It is straightforward to verify that

$$\partial_z J_{\bar{z}}^a = 0. \tag{34}$$

In order to exhibit $SL(2,C)$ invariance of the action (20), it is convenient to identify it with the geometric action for $F_1 \circ F_2^{-1}$. Recall that [11] the geometric action is invariant under $SL(2,C)$ transformations:

$$F_1 \circ F_2^{-1} \xrightarrow{SL(2,C)} \frac{a(\bar{z}) F_1 \circ F_2^{-1} + b(\bar{z})}{c(\bar{z}) F_1 \circ F_2^{-1} + d(\bar{z})}, \tag{35}$$

with $ad - bc = 1$. Noether currents for these transformations, whose infinitesimal form we write as

$$\delta_{SL(2,C)} F_1 \circ F_2^{-1} = \varepsilon_-(\bar{z}) + \varepsilon_0(\bar{z}) F_1 \circ F_2^{-1}$$
$$+ \varepsilon_+(\bar{z})(F_1 \circ F_2^{-1})^2, \tag{36}$$

are precisely those given in (31) to (33).

Next, we discuss the consequences of reparametrization invariance of (20). In the $\bar{z}$-sector, an infinitesimal conformal transformation results in

$$\delta_{\bar{\varepsilon}} F_i = \bar{\varepsilon}(\bar{z}) \partial_{\bar{z}} F_i; \quad (i = 1, 2). \tag{37}$$

Expressing the change in Γ_{eff} as

$$\delta_{\bar{\varepsilon}} \Gamma_{\text{eff}}(F_1, F_2) = \frac{1}{2\pi} \int dz \wedge d\bar{z}\, \bar{\varepsilon}(\bar{z}) \partial_z T_{\bar{z}\bar{z}}, \tag{38}$$

we find

$$T_{\bar{z}\bar{z}}(F_1, F_2) = T_{\bar{z}\bar{z}}(F_1) + T_{\bar{z}\bar{z}}(F_2) + T_{\bar{z}\bar{z}}^{\text{Int}}(F_1, F_2), \tag{39}$$

where,

$$T_{\bar{z}\bar{z}}(F_1) = -\frac{k}{2}\left\{\frac{(\partial_{\bar{z}} F_1)(\partial_z^2 \partial_{\bar{z}} F_1)}{(\partial_z F_1)^2} - \frac{1}{2}\left(\frac{\partial_{\bar{z}} \partial_z F_1}{\partial_z F_1}\right)^2\right.$$
$$\left. - \frac{\partial_z^2 F_1 \partial_{\bar{z}} F_1 \partial_z \partial_{\bar{z}} F_1}{(\partial_z F_1)^2}\right\}, \tag{40}$$

$$T_{\bar{z}\bar{z}}(F_2) = \frac{k}{2}\left[\left(\partial_z\left(\frac{\partial_{\bar{z}} F_2}{\partial_z F_2}\right)\right)^2 - 2\left(\frac{\partial_{\bar{z}} F_2}{\partial_z F_2}\right)\partial_z^2\left(\frac{\partial_{\bar{z}} F_2}{\partial_z F_2}\right)\right], \tag{41}$$

and

$$T_{\bar{z}\bar{z}}^{\text{Int}}(F_1, F_2) = \frac{k}{2}\left\{\left(\frac{\partial_{\bar{z}} F_1}{\partial_z F_1}\right)\left[\partial_z\left(\frac{\partial_{\bar{z}} F_2}{\partial_z F_2}\right)\frac{\partial_z^2 F_1}{\partial_z F_1}\right.\right.$$
$$\left. + \left(\frac{\partial_{\bar{z}} F_2}{\partial_z F_2}\right)\left(\frac{\partial_z^3 F_1}{\partial_z F_1} - \left(\frac{\partial_z^2 F_1}{\partial_z F_1}\right)^2\right) + \partial_z^2\left(\frac{\partial_{\bar{z}} F_2}{\partial_z F_2}\right)\right]$$
$$- \frac{\partial_z \partial_{\bar{z}} F_1}{\partial_z F_1}\left[2\frac{\partial_{\bar{z}} F_2}{\partial_z F_2}\left(\frac{\partial_z^2 F_1}{\partial_z F_1}\right) - \partial_z\left(\frac{\partial_{\bar{z}} F_2}{\partial_z F_2}\right)\right]$$
$$+ \frac{\partial_z^2 \partial_{\bar{z}} F_1}{\partial_z F_1}\left(\frac{\partial_{\bar{z}} F_2}{\partial_z F_2}\right) - \left(\frac{\partial_{\bar{z}} F_2}{\partial_z F_2}\right)^2 D_z F_1\right\}. \tag{42}$$

(40) and (41) can be recognized to be the corresponding $T_{\bar{z}\bar{z}}$ for the geometric and light-cone action respectively.

It is readily verified that, using the equation of motion (21),

$$\partial_z T_{\bar{z}\bar{z}}(F_1, F_2) = 0. \tag{43}$$

We expect $T_{\bar{z}\bar{z}}(F_1, F_2)$ to be of Sugawara form, and indeed, it can be verified that

$$T_{\bar{z}\bar{z}}(F_1, F_2) = -\frac{1}{k} \eta_{ab} J_{\bar{z}}^a J_{\bar{z}}^b, \tag{44}$$

where

$$\eta_{00} = 1; \quad \eta_{1-1} = \eta_{-11} = -\frac{1}{2}. \tag{45}$$

We thus conclude that the induced 2-d extrinsic gravity theory on $h\sqrt{g} = 1$ surfaces in R^3 has conformal invariance in the $\bar{z}$-sector. Furthermore, this conformal field theory is in the same universality class as the induced 2-d intrinsic gravity theory.

Let us briefly discuss the energy-momentum tensor $T_{zz}(F_1, F_2)$. Under infinitesimal conformal change

$$\delta_{\varepsilon(z)} F_i(\bar{z}, z) = \varepsilon(z) \partial_z F_i(\bar{z}, z); \quad (i = 1, 2). \tag{46}$$

This leads to

$$\delta_\varepsilon \Gamma_{\text{eff}}(F_1, F_2) = \frac{1}{2\pi} \int \varepsilon(z) \nabla_{\bar{z}}(F_2) T_{zz}, \tag{47}$$

where

$$\tag{48}$$
$$\nabla_{\bar{z}}(F_2) T_{zz} = \partial_{\bar{z}} T_{zz} - \left(\frac{\partial_{\bar{z}} F_2}{\partial_z F_2}\right)\partial_z T_{zz} - 2\partial_z\left(\frac{\partial_{\bar{z}} F_2}{\partial_z F_2}\right) T_{zz},$$

and

$$T_{zz}(F_1, F_2) = \frac{k}{2}[D_z F_1 - D_z F_2]. \tag{49}$$

From the equation of motion, one finds that

$$\nabla_{\bar{z}}(F_2) T_{zz}(F_1, F_2) = 0. \tag{50}$$

The transformation of the $SL(2, C)$ currents and $T_{\bar{z}\bar{z}}(F_1, F_2)$ under conformal and $SL(2, C)$ transformations can be easily worked out. We give below the results for completeness and will be used in quantizing the theory in the next section

(i) SL(2, C) Variations:

$$\delta_{SL(2, C)} J_{\bar{z}} = -\frac{k}{2}(\partial_{\bar{z}}\varepsilon) - [\varepsilon, J_{\bar{z}}], \tag{51}$$

where

$$J_{\bar{z}} = \eta_{ab} T^a J_{\bar{z}}^a, \tag{52}$$

$$\varepsilon = \varepsilon_a(\bar{z}) T^a, \tag{53}$$

$$T^0 = \frac{1}{2}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad T^+ = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad T^- = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}, \tag{54}$$

$$[T^a, T^b] = f_c^{ab} T^c, \tag{55}$$

$$f_0^{1-1} = -2, \quad f_1^{01} = -f_{-1}^{0-1} = 1. \tag{56}$$

(ii) Conformal Variations:

$$\delta_{\bar{\varepsilon}(\bar{z})} J_{\bar{z}}^a = \bar{\varepsilon}(\bar{z})\partial_z J_{\bar{z}}^a + (\partial_{\bar{z}}\bar{\varepsilon}) J_{\bar{z}}^a, \quad (a = \pm, 0), \tag{57}$$

and

$$\tag{58}$$
$$\delta_{\bar{\varepsilon}(\bar{z})} T_{\bar{z}\bar{z}}(F_1, F_2) = \bar{\varepsilon}(\bar{z})\partial_{\bar{z}} T_{\bar{z}\bar{z}}(F_1, F_2) + 2(\partial_{\bar{z}}\bar{\varepsilon}) T_{\bar{z}\bar{z}}(F_1, F_2).$$

(51) and (57) follow from (31)–(33) and (43) using

$$\delta_{\bar{\varepsilon}(\bar{z})} F_i = \bar{\varepsilon}(\bar{z})\partial_{\bar{z}} F_i, \quad (i = 1, 2). \tag{59}$$

Before quantizing the induced extrinsic gravity theory in the $\bar{z}$-sector, we must first modify the conformal properties of F_1. If we regard Γ_{eff} in (20) as equivalent to the geometric action for $F_1 \cdot F_2^{-1} \equiv F_1(\bar{z}, F_2^{-1}(\bar{z}, z))$, then following Aoyoma [12], one should modify the conformal weight of $F_1 \cdot F_2^{-1}$ in the $\bar{z}$-sector, such that

$$\tag{60}$$
$$\delta_{\bar{\varepsilon}(\bar{z})}(F_1 \cdot F_2^{-1}) = \bar{\varepsilon}(\bar{z})\partial_{\bar{z}}(F_1 \cdot F_2^{-1}) + (\partial_{\bar{z}}\bar{\varepsilon})(F_1 \cdot F_2^{-1}).$$

(60) which assigns conformal spin 1 for F_1 in $\bar{z}$-sector, is necessitated by the equivalence of the geometric action with the Liouville theory.

The same conclusion can also be arrived at looking from the point of view of the light-cone action. Following [10], let us observe that (20) expressed as $\Gamma_+(F_2 \cdot F_1^{-1})$ is the gauge fixed form of the action on $h\sqrt{g} = 1$ surfaces, in the gauge, $\frac{\delta\Gamma_+}{\delta A_{\bar{z}}^0} = 0$, $\frac{\delta\Gamma_+}{\delta A_{\bar{z}}^+} = 1$ (see (16)). The quantum action $S_+(F_2 \cdot F_1^{-1})$ which

guarantees general covariance will contain a term proportional to $\int \left(\frac{\delta\Gamma_+}{\delta A_{\bar{z}}}\right) T_{\bar{z}\bar{z}}$. This has been shown in [10] to yield an extra term to $T_{\bar{z}\bar{z}}$ which is $\partial_{\bar{z}} J_{\bar{z}}^0$. This is equivalent to the modification given in terms of F_1 and F_2 as

$$\delta_{\bar{\varepsilon}(\bar{z})} F_1 = \bar{\varepsilon}(\bar{z})\partial_{\bar{z}} F_1 + (\partial_{\bar{z}}\bar{\varepsilon}) F_1,$$
$$\delta_{\bar{\varepsilon}(\bar{z})} F_2 = \bar{\varepsilon}(\bar{z})\partial_{\bar{z}} F_2. \tag{61}$$

Under this modification, (43) still holds with the modified energy-momentum tensor

$$T_{\bar{z}\bar{z}}^{\text{mod}}(\bar{z}) = -\frac{1}{k}\eta^{ab} J_{\bar{z}}^a J_{\bar{z}}^b - \partial_{\bar{z}} J_{\bar{z}}^0. \tag{62}$$

Under the modified conformal transformations, the conformal weight of the currents also change. We find instead of (57)

$$\delta_{\bar{\varepsilon}(\bar{z})} J_{\bar{z}}^a = (\bar{\varepsilon}(\bar{z})\partial_{\bar{z}} + (1 + a)\partial_{\bar{z}}\bar{\varepsilon}(\bar{z})) J_{\bar{z}}^a, \tag{63}$$

while, (58) remains valid for $T_{\bar{z}\bar{z}}^{\text{mod}}(F_1, F_2)$.

4. Quantum Theory on $h\sqrt{g} = 1$ Surfaces Immersed in R^3

We have all the machinery in place for quantizing our theory of $h\sqrt{g} = 1$ surfaces in R^3. The $\bar{z}$-sector of the action (20) can be quantized like the WZNW action [13]. The details of this procedure are hardly new and we are contend with merely summarizing the essential steps. From (31) to (33), we can expand $J_{\bar{z}}^a$ in Laurent series as

$$J_{\bar{z}}^a = \sum_{n=-\infty}^{\infty} J_n^a \bar{z}^{-n-1-a}, \tag{64}$$

while from (62), $T_{\bar{z}\bar{z}}^{\text{mod}}$ is expanded as

$$T_{\bar{z}\bar{z}}^{\text{mod}} = \sum_{n=-\infty}^{\infty} L_n \bar{z}^{-n-2}. \tag{65}$$

Then, consistent with the $SL(2, C)$ transformation (51), the currents are quantized as (see also [14])

$$[J_n^a, J_m^b] = -\frac{k}{2}(n + a)\eta^{ab}\delta_{n+m, 0} + f_c^{ab} J_{n+m}^c, \tag{66}$$

with

$$J_n^a|0> = 0, \tag{67}$$

for $n > -a$. Next, we define the quantum version of the energy-momentum tensor $T_{\bar{z}\bar{z}}$ or L_n. As shown in

[13], it is given by

$$L_n = \frac{1}{(k+2)} \sum_{m=-\infty}^{\infty} \eta_{ab} : J_m^a J_{n-m}^b : + (n+1) J_n^0 , \quad (68)$$

where the normal ordering (::) is understood as

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad (69)$$

$$\sum_{m=-\infty}^{\infty} : J_m^a J_{n-m}^b : = \sum_{m=-\infty}^{(n+b)} J_m^a J_{n-m}^b + \sum_{(m=n+b+1)}^{\infty} J_{n-m}^b J_m^a .$$

We are now in a position to calculate the rest of the commutators. We find [12]

$$[L_n, J_m^a] = -(m-an) J_{n+m}^a - \frac{k}{2} n(n+1) \eta^{a0} \delta_{n+m} , \quad (70)$$

and

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad (71)$$

$$[L_n, L_m] = (n-m) L_{n+m} + \frac{c_L}{12} n(n-1)(n+1) \delta_{n+m,0} ,$$

where,

$$c_L = \frac{3k}{k+2} - 6k$$

$$= 15 - 6(k+2) - \frac{6}{k+2} . \quad (72)$$

Note that $-6k$ comes from the modified term in $T_{\bar z \bar z}$. Thus the central charge of the Virasaro algebra in the left sector on $h\sqrt{g}=1$ surfaces is the *same* as for the 2-d intrinsic gravity in the same sector.

It is interesting to note that in [5], we observed that conformally immersed surfaces in R^3 with $h\sqrt{g}=1$ exhibit Virasaro invariance with $\frac{H_{zz}}{\sqrt{g}} \equiv D_z F_1$ transforming like the energy-momentum tensor and yet, it is the $\bar z$-sector of this theory which can be successfully quantized. For the z-sector because of (50), the central charge remains unknown.

5. Conclusions

In this article we have studied the classical and quantum properties of a class of conformally immersed surfaces in background R^3. The class of surfaces is characterized by their extrinsic geometric property; i.e., $h\sqrt{g}=1$. We have constructed an action on such surfaces which depends on $\frac{H_{zz}}{\sqrt{g}}$ and $H_{\bar z \bar z}$ as dynamical field variables. Here H_{zz} and $H_{\bar z \bar z}$ are

components of the second fundamental form of the surface and g is the determinant of the induced metric on the surface. The construction of the extrinsic geometric WZNW action is done by minimally coupling 2-d fermions to gauge fields on the surface, constructed from the structure equations to the surface. We show that the gauge restriction $h\sqrt{g}=1$ has an explicit representation which is equivalent to a gauge invariant coupling of light-cone and geometric type actions studied in the literature in connection with intrinsic 2-d gravity theory.

We emphasize in this article the importance of the role of the Gauss map in establishing the existence of Virasaro symmetry in $h\sqrt{g}=1$ surfaces. The Gauss map of conformally immersed surfaces in R^n, studied in detail elsewhere, is the map from a surface in the Grassmannian manifold $G_{2,n}$. Since, not every Grassmannian manifold field forms tangent planes to a surface in R^n, $(n-2)$ integrability conditions must be satisfied by the $G_{2,n}$ fields. The effective action $\Gamma_{\text{eff}}(F_1, F_2)$ is a function of two complex functions F_1, and F_2 which parameterize $H_{zz}/\sqrt{g} = D_z F_1$, and $H_{\bar z \bar z} = \frac{\partial_{\bar z} F_2}{\partial_z F_2}$ and are shown to be the image of X^μ in $G_{2,3}$. The effective action $\Gamma_{\text{eff}}(F_1, F_2) = \Gamma_-(F_1 \cdot F_2^{-1}) = \Gamma_+(F_2 \cdot F_1^{-1})$ is shown to be the Grassmannian σ-model action for induced extrinsic curvature action. Because of a useful composition formula for $\Gamma_{\text{eff}}(F_1, F_2)$, it is found that the induced 2-d extrinsic gravity theory in R^3 on $h\sqrt{g}=1$ surfaces is in the same universality class as the intrinsic 2-d gravity theory. This result constitutes the main observation in this work. Quantization of $h\sqrt{g}=1$ surfaces is carried out with the $SL(2,C)$ currents in the $\bar z$-sector of the theory.

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